Westlake University Graduate Mathematics Admissions Examination Syllabus

2024-2025

1. Algebra and geometry

1.1 Set and logic

(1) Propositional logic, predicate logic.
(2) Sets, correspondences, mappings, injections, surjections, bijections.
(3) Images and preimages of mappings, inverse mappings.
(4) Cartesian product of sets, disjoint union, power set of a set.
(5) Binary relations, reflexivity, symmetry, transitivity, equivalence relations, equivalence classes, quotient sets.
(6) Partially ordered sets, total order, well order, intervals, mathematical induction.
(7) Ordinals, cardinals, countable sets.
(8) Upper bounds, lower bounds, maximal elements, minimal elements, greatest element, least element, supremum, infimum.
(9) Axiom of Choice and its equivalent forms, Zorn's Lemma.

1.2 Groups

(1) Composition laws, semigroups, monoids, invertible elements.
(2) Groups, Abelian groups, group homomorphisms, direct products of groups.
(3) Subgroups, subgroups generated by a subset, order of an element, normal subgroups, quotient groups.
(4) Group actions, stabilizer subgroups, quotient sets, conjugacy classes.
(5) Cyclic groups, finite Abelian groups.
(6) Permutation groups, transpositions, cycles, sign of a permutation, al-

ternating groups.

1.3 Rings

(1) Unitary rings, ring homomorphisms, subrings, product of rings, two-sided ideals, quotient rings.

(2) Ideals in commutative unitary rings, prime ideals, maximal ideals, Chinese remainder theorem.

(3) Algebras over commutative unitary rings, single or multivariable polynomial algebras, algebra of formal power series, polynomial division with remainder, roots of polynomials with invertible leading coefficients, multiplicity, relationships between roots and coefficients of monic splitting polynomials, homogeneous polynomials, symmetric polynomials.

(4) Division rings, fields, fields of fractions of an integral domain, subfields, field characteristics, Frobenius homomorphisms, field extensions.

(5) Integral domains, principal ideal domains, Euclidean domains, prime elements, irreducible elements, invertible elements, greatest common divisor, least common multiple, unique factorization domains, polynomial algebras over unique factorization domains, finitely generated modules over principal ideal domains.

(6) Constructions of the fields of rational numbers, real numbers, and complex numbers, fundamental theorem of algebra.

(7) Finite modules over commutative unitary rings, integral elements in an algebra, algebraic and transcendental extensions of fields, algebraically closed fields, residue fields of irreducible polynomials, splitting fields, cyclotomic fields, separable extensions, normal extensions, Galois group.

(8) Arithmetic of the ring of integers and polynomial rings over fields, Eisenstein's criterion, invertible elements in quotient rings, Euler's function.

1.4 Linear algebra

(1) Modules, module homomorphisms, direct products and direct sums of modules, direct sum decomposition, submodules, quotient modules,

system of generators, free module, basis.

(2) Kernel of a module homomorphism, isomorphism theorems.

(3) Dual module, module composed of module homomorphisms, dual of a module homomorphism, algebra of module endomorphisms, matrices, transpose of matrices, matrices representing homomorphisms between finitely generated free modules.

(4) Vector spaces over fields, linear mappings, existence of bases.

(5) Invariant subspaces of vector space automorphisms, eigenvalues, eigenvectors, eigenspaces, polynomials of linear automorphisms, kernel lemma.

(6) Bases of finitely generated vector spaces, dimension, rank of linear mappings and matrices, echelon form matrices, solutions to systems of linear equations, equations determining linear subspaces.

(7) Multilinear mappings, symmetry and alternation, tensor products, exterior products, determinant and trace of linear automorphisms, determinantal criterion for invertibility, general linear group and special linear group, orientation of real vector spaces.

(8) Characteristic polynomial of a linear endomorphism, minimal polynomial, Cayley-Hamilton theorem, diagonalization, nilpotent automorphisms, Dunford decomposition, Jordan canonical form.

1.5 Bilinear algebra

(1) Bilinear forms, symmetry, kernel of a bilinear form, non-degenerate bilinear forms, quadratic forms, rank of quadratic forms, polarization of quadratic forms, parallelogram equality, matrix representation of bilinear forms, adjoint mapping of a linear mapping.

(2) Orthogonality, isotropic cones, isotropic subspaces, decomposition of bilinear forms into sums of squares, Sylvester's law of inertia, classification of quadratic forms over $\mathbb{R}$ and over $\mathbb{C}$, orthogonalization.

(3) Sesquilinear forms on complex vector spaces, Hermitian property, semi-inner product, Cauchy-Schwarz inequality, semi-norm, inner product, norm, Hilbert spaces.

(4) Orthogonal group, special orthogonal group, decomposition of orthogonal automorphisms into compositions of reflections, unitary group, adjoint mapping of a linear mapping, self-adjoint operators, normal

operators, diagonalization of normal operators, simultaneous reduction of two real quadratic forms, Euclidean geometry of planes and three-dimensional spaces, cross product of vectors in three-dimensional vector space.

1.6 Euclidean geometry

(1) Affine spaces, affine subspaces, barycenter, affine frames, equations of affine spaces.
(2) Affine group, homothety group, angles in affine spaces.
(3) Isometries of affine subspaces in Euclidean spaces, isometry groups.
(4) Convex sets, convex hulls, convex polytopes.
(5) Classification of quadratic hypersurfaces, conic sections in affine planes, quadratic surfaces in three-dimensional spaces.

2. Analysis and probability

2.1 Numerical sequences

(1) Sequences in sets, series in Abelian groups, Abel's summation formula, subsequences.
(2) Extended real line, sequential completeness, Cauchy sequences, upper limits, lower limits, limits, Bolzano-Weierstrass theorem.
(3) Numerical sequences, numerical series, alternating series, series with positive terms, absolute convergence, geometric series, Riemann series, comparison of series and integrals, estimation of remainders, products of series.

2.2 Topology

(1) Metric spaces, open sets, closed sets, bounded sets, topology, topological spaces, induced topology, product of topological spaces.
(2) Accumulation points, closure, limits of sequences and mappings, continuous mappings, homeomorphisms.
(3) Compactness, sequential compactness, connectedness, connected components, path connectedness.
(4) Lipschitz mappings, uniform continuity, Heine's theorem.
(5) Completeness of metric spaces, Baire spaces, fixed point theorem for contraction mappings.
(6) Normed vector spaces, equivalence of norms, finite-dimensional cases, Banach spaces, Hahn-Banach theorem, ℓ^p spaces, absolutely convergent series in Banach spaces.

(7) Bounded linear mappings, operator norms, uniform convergence norms, Arzela-Ascoli theorem.
(8) Inner product spaces, Hilbert spaces, orthogonal projections on closed subspaces.
(9) Dual of inner product spaces, Riesz representation theorem, orthonormal bases, orthogonal polynomials.

2.3 Differential calculus

(1) Intermediate value theorem for continuous univariate functions, continuity of monotonic functions, continuity of inverse functions.
(2) Differentiability, differentiation of composite functions, differentiation of inverse functions, mean value inequality, mean value theorem for univariate functions.
(3) Partial derivatives, higher order derivatives, Taylor's formula, integral remainder, Taylor series expansion of functions.
(4) Sequences of function terms, series of function terms, absolute convergence, uniform convergence, normal convergence.
(5) Continuity, differentiability of limits of sequences of function terms.
(6) Jacobian, Hessian, local extrema, convex functions, criteria for convexity.
(7) Diffeomorphisms, inverse function theorem, implicit function theorem.

2.4 Integral calculus

(1) σ-algebra, measure space, Borel σ-algebra, measure, Lebesgue measure, product measure space, measurable mappings.
(2) Integration of non-negative measurable functions, monotone convergence theorem, Fatou's lemma, dominated convergence theorem.
(3) Integration of vector-valued functions, integrals with parameters, continuity and differentiability with respect to parameters.
(4) L^p spaces, Hölder's inequality, completeness.
(5) Fubini's theorem, substitution formula for integrals, substitution in polar and spherical coordinates, integration by parts.
(6) Convolution, approximation by smooth functions.

(7) Fourier series, Riemann-Lebesgue lemma, Dirichlet theorem, Fejér kernel, Parseval's theorem.
(8) Distributions, Fourier transform, Schwartz space, Plancherel theorem, inverse Fourier transform.

2.5 Differential equation

(1) Ordinary differential equations, initial value problems, Cauchy-Lipschitz theorem, maximal solutions, Grönwall's inequality.
(2) Linear ordinary differential equations, method of variation of constants, higher order linear ordinary differential equations.
(3) Linear partial differential equations, elliptic, hyperbolic, and parabolic partial differential equations, wave equation, heat equation.

2.6 Differential geometry

(1) Submanifolds of Euclidean space, local charts, local parameters, local equations, parametric curves.
(2) Tangent vectors, gradient, tangent space, cotangent space, curvature.
(3) Lagrange multipliers method.

2.7 Complex analysis

(1) Power series, radius of convergence, continuity, complex differentiability, differential equations of power series.
(2) Complex exponentials, power series expansions of common functions.
(3) Holomorphic functions, Cauchy-Riemann equations, path integrals, complex logarithm, integrals with complex parameters.
(4) Cauchy's theorem, meromorphic functions, Cauchy integral formula, analyticity of holomorphic functions, Laurent series for meromorphic functions, residue theorem, separation of poles and zeros, maximum modulus principle, analytic continuation, uniqueness of analytic continuation.

(5) Sequences and series of holomorphic functions, holomorphy of uniform convergent limits.

2.8 Probability

(1) Random events, random variables, probability measures, probability distributions, independence, joint distributions, distribution functions, density functions.
(2) Bernoulli distribution, binomial distribution, geometric distribution, hypergeometric distribution, Poisson distribution, uniform distribution, exponential distribution, normal distribution.
(3) Mathematical expectation, moments, variance, standard deviation, characteristic functions, Laplace transform, sum of independent random variables.
(4) Convergence of random variables, convergence in probability, L^p convergence, almost everywhere convergence.
(5) Markov's inequality, Chebyshev's inequality.
(6) Law of large numbers, central limit theorem.
(7) Radon-Nikodym theorem, conditional probability, conditional expectation.
(8) Statistics, least squares estimation, parameter estimation, hypothesis testing, Fisher information, Cramer-Rao inequality.

2.9 Numerical analysis

(1) Gaussian elimination, LU decomposition of matrices, numerical methods for computing eigenvalues.
(2) Convergence rates, iterative methods, bisection method, Newton's method, error estimation.
(3) Numerical integration, Monte-Carlo methods, numerical calculation of multiple integrals.
(4) Interpolation, Lagrange polynomials, error estimation.
(5) Numerical methods for ordinary differential equations.